COHEN-MACAULAY CHORDAL GRAPHS

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ABSTRACT. We classify all Cohen-Macaulay chordal graphs. In particular, it is shown that a chordal graph is Cohen-Macaulay if and only if it is unmixed.

INTRODUCTION

To each finite graph G with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E(G)$ one associates the edge ideal $I(G) \subset K[x_1, \ldots, x_n]$ which is generated by all monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Here K is an arbitrary field. The graph G is called Cohen-Macaulay over K, if $K[x_1, \ldots, x_n]/I(G)$ is a Cohen-Macaulay ring, and is called Cohen-Macaulay if it is Cohen-Macaulay over any field.

Given a field K. The general problem is to classify the graphs which are Cohen-Macaulay over K. In this generality the problem is as hard as to classify all Cohen-Macaulay simplicial complexes, because given a simplicial complex Δ, one can naturally construct a finite graph G such that G is Cohen–Macaulay if and only if Δ is Cohen–Macaulay. In fact, if P is the face poset of Δ (the poset consisting of all faces of Δ, ordered by inclusion), then Δ is Cohen–Macaulay if and only if the order complex $\Delta(P)$ of P is Cohen–Macaulay. Since the order complex $\Delta(P)$ is flag, i.e., every minimal non-face is a 2-element subset, it follows that there is a finite graph G such that $I(G)$ coincides with the Stanley–Reisner ideal of $\Delta(P)$.

Thus one cannot expect a general classification theorem. On the other hand, the first positive result was given by Villarreal [4] who determined all Cohen-Macaulay trees. This result has been recently widely generalized in [2] where all bipartite Cohen-Macaulay graphs have been described. It turned out that the Cohen-Macaulay property of a bipartite graph does not depend on the field K.

In this note we classify all Cohen-Macaulay chordal graphs. Again it turns out that for chordal graphs the Cohen-Macaulay property is independent of the field K. Indeed we show that G is Cohen-Macaulay if and only if the edge ideal $I(G)$ is height unmixed. One of our tools is Dirac's theorem [1] in a version as presented in [3].

1. PRELIMINARIES

Let G be a finite graph on $[n]$ without loops, multiple edges and isolated vertices, and $E(G)$ its edge set. The graph G is called *chordal* if all cycles of length > 3 has a chord.

A *stable subset* or *clique* of G is a subset F of $[n]$ such that $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. We write $\Delta(G)$ for the simplicial complex on $[n]$ whose faces are the stable

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subsets of G. For the proof of our main theorem we need the following property of chordal graphs [3, Lemma 3.1] which is related to Dirac's theorem [1].

Lemma 1.1. *Let G be a chordal graph. Then $\Delta(G)$ is a quasi-forest.*

We recall the definition of a quasi-forest introduced in [5]: let Δ be a simplicial complex, and $\mathscr{F}(\Delta)$ the set of its facets. A facet $F \in \mathscr{F}(\Delta)$ is called a *leaf*, if there exists a facet G (called a *branch* of F) with $G \neq F$ and such that $H \cap F \subset G \cap F$ for all $H \in \mathscr{F}(\Delta)$ with $H \neq F$. We say that Δ is a *quasi-forest*, if there exists an order $F_1, \ldots, F_r$ of the facets of Δ such that for each $i = 1, \ldots, r$, F_i is a leaf of the simplicial complex $\langle F_1, \ldots, F_i \rangle$ (whose facets are $F_1, \ldots, F_i$).

Let K be a field. A graph G is called *Cohen-Macaulay over K* if the edge ideal $I(G) = (\{x_i x_j \colon \{i, j\} \in E(G)\})$ of G is a Cohen-Macaulay ideal in $S = K[x_1, \ldots, x_n]$, in other words, if $S/I(G)$ is Cohen-Macaulay.

Suppose G is Cohen-Macaulay over K. Then we say *G is of type r* over K, if r is the Cohen-Macaulay type of $S/I(G)$, that is, if r is the minimal number of generators of the canonical module of $S/I(G)$. The Cohen-Macaulay type of a Cohen-Macaulay ring R can also be computed as the socle dimension of the residue class ring of R modulo a maximal regular sequence. The ring R is Gorenstein, if and only if the Cohen-Macaulay type of R is 1. We say that *G is Gorenstein over K*, if $S/I(G)$ is Gorenstein over K.

Finally we say that G is *Cohen-Macaulay*, *of type r*, or *Gorenstein*, if G has the corresponding property over any field.

The minimal prime ideals of $I(G)$ correspond to the minimal vertex covers of G. Recall that a *vertex cover* of G is a subset $C \subset [n]$ such that $C \cap \{i, j\} \neq \emptyset$ for all $\{i, j\} \in E(G)$. It is called *minimal* if no proper subset of C is a vertex cover of G. If we denote by $\mathscr{C}(G)$ the set of minimal vertex covers, then the set of ideals $\{(\{x_i \colon i \in C\}) \colon C \in \mathscr{C}(G)\}$ is precisely the set of minimal prime ideals of $I(G)$.

Suppose again that G is Cohen-Macaulay over K. Then the ideal $I(G)$ is height unmixed. Thus all minimal vertex covers of G have the same cardinality.

For the proof of our main theorem we need the following algebraic fact:

Lemma 1.2. *Let R be a Noetherian ring, $S = R[x_1, \ldots, x_n]$ the polynomial ring over R, k an integer with $0 \leq k < n$, and J the ideal $(I_1 x_1, \ldots, I_k x_k, \{x_i x_j\}_{1 \leq i < j \leq n}) \subset S$, where $I_1, \ldots, I_k$ are ideals in R. Then the element $x = \sum_{i=1}^n x_i$ is a non-zerodivisor on S/J.*

Proof. For a subset $T \subset [n]$ we let L_T be the ideal generated by all monomials $x_i x_j$ with $i, j \in T$ and $i < j$, and we set $I_T = \sum_{j \in T} I_j$ and $X_T = (\{x_j\}_{j \in T})$.

It is easy to see that

$$L_T = \bigcap_{\ell \in T} X_{T \setminus \{\ell\}}.$$

Hence we get

$$
\begin{aligned}
J &= (I_1x_1,\ldots,I_kx_k,L_{[n]}) = \bigcap_{T\subset[k]} (I_T,X_{[k]\setminus T},L_{[n]})\\
&= \bigcap_{T\subset[k]} (I_T,X_{[k]\setminus T},L_{[n]\setminus([k]\setminus T)}) = \bigcap_{\substack{T\subset[k]\\ \ell\in[n]\setminus([k]\setminus T)}} (I_T,X_{[k]\setminus T},X_{([n]\setminus([k]\setminus T))\setminus\{\ell\}})\\
&= \bigcap_{\substack{T\subset[k]\\ \ell\in[n]\setminus([k]\setminus T)}} (I_T,X_{[n]\setminus\{\ell\}}).
\end{aligned}
$$

Thus in order to prove that x is a non-zerodivisor modulo J it suffices to show that x is a non-zerodivisor modulo each of the ideals $(I_T,X_{[n]\setminus\{\ell\}})$. To see this we first pass to the residue class ring modulo I_T, and hence if we replace R by R/I_T it remains to be shown that x is a non-zerodivisor on $R[x_1,\ldots,x_n]/(x_1,\ldots,x_{\ell-1},x_{\ell+1},\ldots x_n)$. But this is obviously the case. $\square$

2. The classification

Theorem 2.1. *Let K be a field, and let G be a chordal graph on the vertex set $[n]$. Let $F_1,\ldots,F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following conditions are equivalent:*

(a) *G is Cohen-Macaulay;*
(b) *G is Cohen-Macaulay over K;*
(c) *G is unmixed;*
(d) *$[n]$ is the disjoint union of $F_1,\ldots,F_m$.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): Since any Cohen-Macaulay ring is height unmixed it follows that G is unmixed.

(c) $\Rightarrow$ (d): Let G be a unmixed chordal graph on $[n]$ and $E(G)$ the set of edges of G. Let $F_1,\ldots,F_m$ denote the facets of $\Delta(G)$ with free vertices. Fix a free vertex v_i of F_i and set $W=\{v_1,\ldots,v_m\}$. Suppose that $B=[n]\setminus(\bigcup_{i=1}^m F_i)\neq\emptyset$ and write $G|_B$ for the induced subgraph of G on B. Since $\{v_i,b\}\notin E(G)$ for all $1\le i\le m$ and for all $b\in B$, if X $(\subset B)$ is a minimal vertex cover of $G|_B$, then $X\cup((\bigcup_{i=1}^m F_i)\setminus W)$ is a minimal vertex cover of G. In particular $G|_B$ is unmixed. Since the induced subgraph $G|_B$ is again chordal, by working with induction on the number of vertices, it follows that if $H_1,\ldots,H_s$ are the facets of $\Delta(G|_B)$ with free vertices, then B is the disjoint union $B=\bigcup_{j=1}^s H_j$. Let v'_j be a free vertex of H_j and set $W'=\{v'_1,\ldots,v'_s\}$. Since $((\bigcup_{i=1}^m F_i)\setminus W)\cup(B\setminus W')$ is a minimal vertex cover of G and since G is unmixed, every minimal vertex cover of G consists of $n-(m+s)$ vertices.

We claim that $F_i\cap F_j=\emptyset$ if $i\neq j$. In fact, if, say, $F_1\cap F_2\neq\emptyset$ and if $w\in[n]$ satisfies $w\in F_i$ for all $1\le i\le\ell$, where $\ell\ge 2$, and $w\notin F_i$ for all $\ell<i\le m$, then $Z=(\bigcup_{i=1}^m F_i)\setminus\{w,v_{\ell+1},\ldots,v_m\}$ is a minimal vertex cover of the induced subgraph $G'=G|_{[n]\setminus B}$ on $[n]\setminus B$. Let Y be a minimal vertex cover of G with $Z\subset Y$. Since $Y\cap B$ is a vertex cover of $G|_B$, one has $|Y\cap B|\ge|B|-s$. Moreover, $|Y\cap([n]\setminus B)|\ge n-|B|-(m-\ell+1)>n-|B|-m$. Hence $|Y|>n-(m+s)$, a contradiction.

Consequently, a subset Y of $[n]$ is a minimal vertex cover of G if and only if $|Y \cap F_i| = |F_i| - 1$ for all $1 \leq i \leq m$ and $|Y \cap H_j| = |H_j| - 1$ for all $1 \leq j \leq s$.

Now, since $\Delta(G|_B)$ is a quasi-forest, one of the facets $H_1, \ldots, H_s$ must be a leaf of $\Delta(G|_B)$. Let, say, H_1 be a leaf of $\Delta(G|_B)$. Let δ and δ', where $\delta \neq \delta'$, be free vertices of H_1 with $\{\delta, a\} \in E(G)$ and $\{\delta', a'\} \in E(G)$, where a and a' belong to $[n] \setminus B$. If $a \neq a'$ and if $\{a, a'\} \in E(G)$, then one has either $\{\delta, a'\} \in E(G)$ or $\{\delta', a\} \in E(G)$, because G is chordal and $\{\delta, \delta'\} \in E(G)$. Hence there exists a subset $A \subset [n] \setminus B$ such that

(i) $\{a, b\} \notin E(G)$ for all $a, b \in A$ with $a \neq b$,
(ii) for each free vertex δ of H_1, one has $\{\delta, a\} \in E(G)$ for some $a \in A$, and
(iii) for each $a \in A$, one has $\{\delta, a\} \in E(G)$ for some free vertex δ of H_1.

In fact, it is obvious that a subset $A \subset [n] \setminus B$ satisfying (ii) and (iii) exists. If $\{a, a'\} \in E(G)$, $\{\delta, a\} \in E(G)$ and $\{\delta, a'\} \notin E(G)$ for some $a, a' \in A$ with $a \neq a'$ and for a free vertex δ of H_1, then every free vertex δ' of H_1 with $\{\delta', a'\} \in E(G)$ must satisfy $\{\delta', a\} \in E(G)$. Hence $A \setminus \{a'\}$ satisfies (ii) and (iii). Repeating such the technique yields a subset $A \subset [n] \setminus B$ satisfying (i), (ii) and (iii), as required.

If $s > 1$, then H_1 has a branch. Let $w_0 \notin H_1$ be a vertex belonging to a branch of the leaf H_1 of $\Delta(G|_B)$. Thus $\{\xi, w_0\} \in E(G)$ for all nonfree vertices ξ of H_1. We claim that either $\{a, w_0\} \notin E(G)$ for all $a \in A$, or one has $a \in A$ with $\{a, \xi\} \in E(G)$ for every nonfree vertices ξ of H_1. To see why this is true, if $\{a, w_0\} \in E(G)$ and $\{\delta, a\} \in E(G)$ for some $a \in A$ and for some free vertex δ of H_1, then one has a cycle (a, δ, ξ, w_0) of length four for every nonfree vertex ξ of H_1. Since $\{\delta, w_0\} \notin E(G)$, one has $\{a, \xi\} \in E(G)$.

Let X be a minimal vertex cover of G such that $X \subset [n] \setminus (A \cup \{w_0\})$ (resp. $X \subset [n] \setminus A$) if $\{a, w_0\} \notin E(G)$ for all $a \in A$ (resp. if one has $a \in A$ with $\{a, \xi\} \in E(G)$ for every nonfree vertices ξ of H_1.) Then, for each vertex γ of H_1, there is $w \notin X$ with $\{\gamma, w\} \in E(G)$. Hence $H_1 \subset X$, in contrast to our considerations before. This contradiction guarantees that $B = \emptyset$. Hence $[n]$ is the disjoint union $[n] = \bigcup_{i=1}^m F_i$, as required.

Finally suppose that $s = 1$. Then H_1 is the only facet of $\Delta(G|_B)$. Then $X = \bigcup_{i=1}^m (F_i \setminus v_i)$ is a minimal free vertex cover G with $H_1 \subset X$, a contradiction.

(d) $\Rightarrow$ (c): Let $F_1, \ldots, F_m$ denote the facets of $\Delta(G)$ with free vertices and, for each $1 \leq i \leq m$, write F_i for the set of vertices of F_i. Given a minimal vertex cover $X \subset [n]$ of G, one has $|X \cap F_i| \geq |F_i| - 1$ for all i since F_i is a clique of G. If, however, for some i, one has $|X \cap F_i| = |F_i|$, i.e., $F_i \subset X$, then $X \setminus \{v_i\}$ is a vertex cover of G for any free vertex v_i of F_i. This contradicts the fact that X is a minimal vertex cover of G. Thus $|X \cap F_i| = |F_i| - 1$ for all i. Since $[n]$ is the disjoint union $[n] = \bigcup_{i=1}^m F_i$, it follows that $|X| = n - m$ and G is unmixed, as desired.

(c) and (d) $\Rightarrow$ (a): We know that G is unmixed. Moreover, if $v_i \in F_i$ is a free vertex, then $[n] \setminus \{v_1, \ldots, v_m\}$ is a minimal vertex cover of G. In particular it follows that $\dim S/I(G) = m$.

For $i = 1, \ldots, m$, we set $y_i = \sum_{j \in F_i} x_j$. We will show that $y_1, \ldots, y_m$ is a regular sequence on $S/I(G)$. This then yields that G is Cohen-Macaulay.

Let $F_i = \{i_1, \ldots, i_k\}$, and assume that $i_{\ell+1}, \ldots, i_k$ are the free vertices of F_i. Let $G' \subset G$ be the induced subgraph of G on the vertex set $[n] \setminus \{i_1, \ldots, i_k\}$. Then $I(G) = (I(G'), J_1 x_{i_1}, J_2 x_{i_2}, \ldots, J_\ell x_{i_\ell}, J)$, where $J_j = (\{x_r \colon \{r, i_j\} \in E(G)\})$ for $j = 1, \ldots, \ell$, and where $J = (\{x_{i_r} x_{i_s} \colon 1 \leq r < s \leq k\})$.

Since $[n]$ is the disjoint union of $F_1,\dots,F_m$ it follows that all generators of the ideal $(I(G'),y_1,\dots,y_{i-1})$ belong to $K[\{x_i\}_{i\in[n]\setminus F_i}]$. Thus if we set

$$R = K[\{x_i\}_{i\in[n]\setminus F_i}]/(I(G'),y_1,\dots,y_{i-1}),$$

then

$$(S/I(G))/(y_1,\dots,y_{i-1})(S/I(G)) \cong R[x_{i_1},\dots,x_{i_k}]/(I_1x_{i_1},\dots,I_\ell x_{i_\ell},\{x_{i_r}x_{i_s}\colon 1\le r<s\le k\}),$$

where for each j, the ideal I_j is the image of J_j under the residue class map onto R. Thus Lemma 1.2 implies that y_i is regular on $(S/I(G))/(y_1,\dots,y_{i-1})(S/I(G))$. □

Let G be an arbitrary graph on the vertex set $[n]$. An *independent set of* G is a set $S\subset[n]$ such that $\{i,j\}\not\in E(G)$ for all $i,j\in S$. With this notion we can describe the type of a Cohen-Macaulay chordal graph.

Corollary 2.2. *Let G be a chordal graph, and let $F_1,\dots,F_m$ be the facets of $\Delta(G)$ which have a free vertex. Let i_j be a free vertex of F_j for $j=1,\dots,m$, and let G' be the induced subgraph of G on the vertex set $[n]\setminus\{i_1,\dots,i_m\}$. Then*

(a) *the type of G, is the number of maximal independent subsets of G';*
(b) *G is Gorenstein, if and only if G is a disjoint union of edges.*

Proof. (a) Let $F\subset[n]$ and $S=K[x_1,\dots,x_n]$. We note that if J is the ideal generated by the set of monomials $\{x_ix_j\colon i,j\in F \text{ and } i<j\}$, and $x=\sum_{i\in F}x_i$, then for any $i\in F$ one has that

$$(S/J)/x(S/J)\cong S_i/(\{x_j\colon j\in F, j\neq i\})^2,$$

where $S_i=K[x_1,\dots,x_{i-1},x_{i+1},\dots,x_n]$.

Thus if we factor by a maximal regular sequence as in the proof of Theorem 2.1 we obtain a 0-dimensional ring of the form

$$A=T/(P_1^2,\dots,P_m^2,I(G'')).$$

Here $P_j=(\{x_k\colon k\in F_j,\ k\neq i_j\})$, G'' is the subgraph of G consisting of all edges which do not belong to any F_j, and T is the polynomial ring over K in the set of variables $X=\{x_k\colon k\in[n], k\neq i_j \text{ for all } j=1,\dots,m\}$. It is obvious that A is obtained from the polynomial ring T by factoring out the squares of all variables of T and all x_ix_j with $\{i,j\}\in E(G')$. Therefore A has a K-basis of squarefree monomials corresponding to the independent subsets of G', and the socle of A is generated as a K-vector space by the monomials corresponding to the maximal independent subsets of G'.

(b) If G is a disjoint union of edges, then $I(G)$ is a complete intersection, and hence Gorenstein.

Conversely, suppose that G is Gorenstein. Then A is Gorenstein. Since A a 0-dimensional ring with monomial relations, A is Gorenstein if and only if A is a complete intersection. This is the case only if $E(G')=\emptyset$, in which case G is a disjoint union of egdes. □

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